DEFORMABLE TRANSFORMATIONS OF RIBAUCOUR*

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When a system of spheres involves two parameters, their envelope consists in general of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by the points of contact on the same sphere. In general the lines of curvature on Σ and Σ_1 do not correspond. When they do, we say that Σ is in the relation of a transformation of Ribaucour with Σ_1, and vice versa. For the sake of brevity we call it a *transformation* R.

It is a known property of envelopes of spheres that if S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before the deformation. Ordinarily when S for a transformation R is deformed, the sheets Σ' and Σ_1' of the new envelope are not in the relation of a transformation R. Bianchi† has shown that when S is a surface applicable to a surface of revolution, it is possible to choose spheres so that for every deformation of S the two sheets of the envelope shall be in the relation of a transformation R. From the equations which Bianchi used to prove this result it can be shown that if S undergoes a single deformation so that the sheets of the new envelope are in the relation of a transformation R the conjugate system on the deform of S corresponding to the lines of curvature on the envelope is the same as before deformation. It is known that a conjugate system may be preserved in an infinity of deformations, or one, or none, the latter being the general case. It is the purpose of this paper to determine the transformations R whose surfaces of center admit one deformation into surfaces of center of transformations R.

It is shown that the only surfaces Σ admitting such transformations R have the same spherical representation of their lines of curvature as isothermic surfaces, and that every surface of this type admits such transformations.

The conjugate system on the surface S is a system $2O$, to use the notation of Guichard. A conjugate system remaining conjugate in a single deformation is called a *permanent conjugate system*. The present investigation carries

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† *Differenziale geometria*, vol. 2, p. 117.

with it the determination of all permanent conjugate systems which are $2O$. It shows also that any conjugate system having the same spherical representation as a permanent system $2O$ is of this kind also.

Evidently when one sheet of an envelope of spheres is a point or a plane the transformation is of the type R. Hence a special class of the solutions of our problem include all transformations R whose surfaces of center may be deformed so that in the new position the spheres pass through a point or are tangent to a plane, which we have shown to be respectively transformations D_m of isothermic surfaces,* and transformations E_m of surfaces with isothermal representations of their lines of curvature.†

When our general results are applied to the transformations D_m of isothermic surfaces, we are led to the transformations T_m of these surfaces discovered by Bianchi‡ and expressed intrinsically. The present results give a geometrical basis to these transformations and other observations made recently by Bianchi.§

1. Transformations of Ribaucour

Let Σ be a surface referred to its lines of curvature, u = const., v = const.; x, y, z, its cartesian coördinates; X, Y, Z, the direction-cosines of its normal; ρ_1, ρ_2, its principal radii of normal curvature. These quantities satisfy the following equations of Rodrigues:‖

$$\frac{\partial x}{\partial u} + \rho_1 \frac{\partial X}{\partial u} = 0, \qquad \frac{\partial x}{\partial v} + \rho_2 \frac{\partial X}{\partial v} = 0. \tag{1}$$

Darboux¶ has shown that if λ and μ are any two functions satisfying the equations

$$\frac{\partial \lambda}{\partial u} + \rho_1 \frac{\partial \mu}{\partial u} = 0, \qquad \frac{\partial \lambda}{\partial v} + \rho_2 \frac{\partial \mu}{\partial v} = 0, \tag{2}$$

on the surface S whose coördinates x_0, y_0, z_0 are given by

$$x_0 = x - \frac{\lambda}{\mu} X, \qquad y_0 = y - \frac{\lambda}{\mu} Y, \qquad z_0 = z - \frac{\lambda}{\mu} Z, \tag{3}$$

the parametric curves form a conjugate system, and that the spheres with centers on S and whose radii are given by λ/μ give a transformation R of Σ, and the most general one. We shall now find the expressions for the cartesian coördinates x_1, y_1, z_1 of the transform Σ_1.

* Rendiconti dei Lincei, ser. 5, vol. 24 (1915), p. 351.

† Annals of Mathematics, ser. 2, vol. 17 (1915), pp. 64–71.

‡ Annali di Matematica, ser. 3, vol. 12 (1906), pp. 19–54.

§ Rendiconti dei Lincei, ser. 5, vol. 24 (1915), pp. 377–387.

‖ E., p. 122. A reference of this sort is to the author's *Differential Geometry*.

¶ *Lecons*, vol. 2, p. 383.

If $X', Y', Z'; X'', Y'', Z''$ denote the direction-cosines of the tangents to the curves v = const., u = const. respectively on Σ, we have*

$$X' = \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}, \qquad X'' = \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v},$$

$$\tag{4} \begin{gathered} \frac{\partial X'}{\partial u} = -\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}X'' + \frac{\sqrt{E}}{\rho_1}X, \qquad \frac{\partial X'}{\partial v} = \frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}X'', \\ \frac{\partial X''}{\partial u} = \frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}X', \qquad \frac{\partial X''}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}X' + \frac{\sqrt{G}}{\rho_2}X, \\ \frac{\partial X}{\partial u} = -\frac{\sqrt{E}}{\rho_1}X', \qquad \frac{\partial X}{\partial v} = -\frac{\sqrt{G}}{\rho_2}X''.\dagger \end{gathered}$$

The equation of the sphere of radius λ/μ and center (3) is reducible to

$$\mu \sum (x_1 - x)^2 + 2\lambda \sum X(x_1 - x) = 0,$$

where x_1, y_1, z_1 are current coördinates. The equations obtained by differentiating this equation with respect to u and v are

$$\frac{\partial \lambda}{\partial u}\sum (x_1 - x)^2 + 2\lambda\sqrt{E}\sum X'(x_1 - x) = 0,$$

$$\frac{\partial \lambda}{\partial v}\sum (x_1 - x)^2 + 2\lambda\sqrt{G}\sum X''(x_1 - x) = 0.$$

These three equations may be replaced by

$$\tag{5} x_1 - x = -\frac{1}{\sigma m}(\alpha X' + \beta X'' + \mu X),$$

where m is a constant, and the functions α, β, σ are subject to the conditions

$$\tag{6} \alpha^2 + \beta^2 + \mu^2 = 2m\lambda\sigma,$$

and

$$\frac{\partial \lambda}{\partial u} = \alpha\sqrt{E}, \qquad \frac{\partial \lambda}{\partial v} = \beta\sqrt{G}.$$

Comparing equations (1) and (2), we see that λ is a solution of the equation satisfied by x, y, z, that is, the point equation for Σ. Expressing this condition, we find the expressions for $\partial\alpha/\partial v$ and $\partial\beta/\partial u$ in the fundamental equations (I) below. The third and fourth of equations (I) below follow from the

* E., p. 157.

† Algebraic signs are given to these direction-cosines so that we have

$$\begin{vmatrix} X' & Y' & Z' \\ X'' & Y'' & Z'' \\ X & Y & Z \end{vmatrix} = 1.$$

first two and (2). The functions k and l in the fifth and sixth of equations (I) may be taken as defined by these equations. The expressions for $\partial\alpha/\partial u$ and $\partial\beta/\partial v$ are a consequence of the differentiation of (6). The fundamental system is

$$
\text{(I)}\qquad
\begin{aligned}
&\frac{\partial\lambda}{\partial u} = \sqrt{E}\alpha, \qquad \frac{\partial\lambda}{\partial v} = \sqrt{G}\beta,\\
&\frac{\partial\mu}{\partial u} = -\frac{\sqrt{E}}{\rho_1}\alpha, \qquad \frac{\partial\mu}{\partial v} = -\frac{\sqrt{G}}{\rho_2}\beta,\\
&\frac{\partial \log\sigma}{\partial u} = k\frac{\alpha}{\lambda}, \qquad \frac{\partial \log\sigma}{\partial v} = l\frac{\beta}{\lambda},\\
&\frac{\partial\alpha}{\partial u} = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\beta + \frac{\sqrt{E}}{\rho_1}\mu + m(k+\sqrt{E})\sigma,\\
&\frac{\partial\alpha}{\partial v} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\beta, \qquad \frac{\partial\beta}{\partial u} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\alpha,\\
&\frac{\partial\beta}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha + \frac{\sqrt{G}}{\rho_2}\mu + m(l+\sqrt{G})\sigma,\\
&\frac{\partial k}{\partial v} = l\left[\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{\beta}{\lambda}(k+\sqrt{E})\right],\\
&\frac{\partial l}{\partial u} = k\left[\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} - \frac{\alpha}{\lambda}(l+\sqrt{G})\right].
\end{aligned}
$$

The last two of these equations must be satisfied in order that the conditions of integrability of the derivatives of α and β shall be satisfied; in deriving them we make use of the Gauss and Codazzi equations for Σ, namely

$$
\text{(7)}\qquad
\begin{aligned}
&\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\sqrt{EG}}{\rho_1\rho_2} = 0,\\
&\frac{\partial}{\partial v}\left(\frac{\sqrt{E}}{\rho_1}\right) = \frac{1}{\rho_2}\frac{\partial\sqrt{E}}{\partial v}, \qquad \frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\rho_2}\right) = \frac{1}{\rho_1}\frac{\partial\sqrt{G}}{\partial u}.
\end{aligned}
$$

Equations (I) form a completely integrable system of differential equations and each set of solutions satisfying (6) gives a surface Σ_1, which, as we shall show, is a Ribaucour transform of Σ.

From (5) we have by differentiation

$$
\text{(8)}\qquad \frac{\partial x_1}{\partial u} = -kX_1', \qquad \frac{\partial x_1}{\partial v} = lX_1'',
$$

where

$$
(9) \qquad \begin{aligned} X_1' &= \left(1 - \frac{\alpha^2}{m\sigma\lambda}\right) X' - \frac{\alpha\beta}{m\sigma\lambda} X'' - \frac{\alpha\mu}{m\sigma\lambda} X, \\ X_1'' &= \frac{\alpha\beta}{m\sigma\lambda} X' - \left(1 - \frac{\beta^2}{m\sigma\lambda}\right) X'' + \frac{\beta\mu}{m\sigma\lambda} X. \end{aligned}
$$

Similar equations hold for the y's and z's. Now X_1', Y_1', Z_1'; X_1'', Y_1'', Z_1'' are the direction-cosines of the tangents to the curves v = const., u = const. respectively on Σ_1. From these we find that the direction-cosines, X_1, Y_1, Z_1, of the normal to Σ_1 are of the form

$$
(10) \qquad X_1 = -\frac{\alpha\mu}{m\sigma\lambda} X' - \frac{\beta\mu}{m\sigma\lambda} X'' + \left(1 - \frac{\mu^2}{m\sigma\lambda}\right) X.
$$

If E_1 and G_1 are the first fundamental coefficients of Σ_1, we have on comparing (8) with (4) that

$$
(11) \qquad \sqrt{E_1} = -k, \qquad \sqrt{G_1} = l.
$$

On differentiating (10) we get

$$
\frac{\partial X_1}{\partial u} + \frac{1}{\rho_{11}} \frac{\partial x_1}{\partial u} = 0, \qquad \frac{\partial X_1}{\partial v} + \frac{1}{\rho_{12}} \frac{\partial x_1}{\partial v} = 0,
$$

where ρ_{11} and ρ_{12} are given by

$$
(12) \qquad \begin{aligned} k\left(\frac{1}{\rho_{11}} + \frac{\mu}{\lambda}\right) + \sqrt{E}\left(\frac{1}{\rho_1} + \frac{\mu}{\lambda}\right) &= 0, \\ l\left(\frac{1}{\rho_{12}} + \frac{\mu}{\lambda}\right) + \sqrt{G}\left(\frac{1}{\rho_2} + \frac{\mu}{\lambda}\right) &= 0. \end{aligned}
$$

Since the preceding equations are analogous to (1), we know that the lines of curvature on Σ_1 are parametric, and that ρ_{11} and ρ_{12}, given by (12), are the principal radii of normal curvature of Σ_1. Hence

THEOREM 1. *Every set of solutions of equations* (I) *which satisfy the quadratic relation* (6) *determines a transformation* R *of* Σ.

In consequence of (10) equation (5) can be given the form

$$
(13) \qquad x_1 - x = \frac{\mu}{\lambda}(X_1 - X).
$$

2. The transformation R obtained by the deformation of S

From (I) we find

$$
(14) \qquad \frac{\partial}{\partial u}\left(\frac{\lambda}{\mu}\right) = \frac{\alpha}{\mu} L_1, \qquad \frac{\partial}{\partial v}\left(\frac{\lambda}{\mu}\right) = \frac{\beta}{\mu} L_2,
$$

where we have put

$$(15) \qquad L_1 = \sqrt{E}\left(1 + \frac{1}{\rho_1}\frac{\lambda}{\mu}\right), \qquad L_2 = \sqrt{G}\left(1 + \frac{1}{\rho_2}\frac{\lambda}{\mu}\right).$$

Hence the first fundamental coefficients of S, whose coördinates are given by (3), have the form

$$(16) \quad E_0 = L_1^2\left(1 + \left(\frac{\alpha}{\mu}\right)^2\right), \qquad F_0 = L_1 L_2 \frac{\alpha}{\mu}\frac{\beta}{\mu}, \qquad G_0 = L_2^2\left(1 + \left(\frac{\beta}{\mu}\right)^2\right).$$

When S is deformed into the surface of centers S' of another transformation R, the functions of the latter must necessarily give equivalent expressions for E_0, F_0, and G_0. Since the spheres are unaltered in size, the functions λ' and μ' are given by

$$(17) \qquad \lambda' = \rho\lambda, \qquad \mu' = \rho\mu,$$

where ρ is a factor of proportionality.*

We call Σ' and Σ_1' the two sheets of the new transformation R and indicate by primes all functions belonging to the latter. From (14) and analogous equations we have

$$(18) \qquad \begin{aligned} \sqrt{E}\frac{\alpha}{\mu}\left(1 + \frac{1}{\rho_1}\frac{\lambda}{\mu}\right) &= \sqrt{E'}\frac{\alpha'}{\mu'}\left(1 + \frac{1}{\rho_1'}\frac{\lambda}{\mu}\right), \\ \sqrt{G}\frac{\beta}{\mu}\left(1 + \frac{1}{\rho_2}\frac{\lambda}{\mu}\right) &= \sqrt{G'}\frac{\beta'}{\mu'}\left(1 + \frac{1}{\rho_2'}\frac{\lambda}{\mu}\right). \end{aligned}$$

In order that the expressions (16) may be equal to similar ones for S', we must have in consequence of (18)

$$\frac{\alpha'^2}{\mu'^2} = \frac{\alpha^2}{\mu^2}, \qquad \frac{\beta'^2}{\mu'^2} = \frac{\beta^2}{\mu^2}.$$

Because of (17) these may be replaced by

$$(19) \qquad \alpha' = \rho\alpha, \qquad \beta' = -\rho\beta.\dagger$$

When these values are substituted in equations analogous to the first two of (I), we find for the determination of ρ

$$(20) \qquad \frac{\partial \log \rho}{\partial u} = \frac{\alpha}{\lambda}(\sqrt{E'} - \sqrt{E}), \qquad \frac{\partial \log \rho}{\partial v} = -\frac{\beta}{\lambda}(\sqrt{G'} + \sqrt{G}).$$

* By thus fixing the signs of λ' and μ' we have merely chosen the positive direction of the normal to Σ'.

† The choice of these signs is merely equivalent to determining the signs of the square roots of E' and G' hereafter.

Moreover, equations (18) are reducible to

$$(21)\qquad \begin{aligned} \sqrt{E'}\left(1+\frac{1}{\rho_1'}\frac{\lambda}{\mu}\right) &= \sqrt{E}\left(1+\frac{1}{\rho_1}\frac{\lambda}{\mu}\right), \\ -\sqrt{G'}\left(1+\frac{1}{\rho_2'}\frac{\lambda}{\mu}\right) &= \sqrt{G}\left(1+\frac{1}{\rho_2}\frac{\lambda}{\mu}\right). \end{aligned}$$

If these equations be differentiated with respect to v and u respectively and in the reduction use be made of the Codazzi equations (7) and similar ones for Σ', we get

$$(22)\qquad \begin{aligned} \frac{1}{\sqrt{G'}}\frac{\partial\sqrt{E'}}{\partial v}+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} &= -\frac{\beta}{\lambda}(\sqrt{E'}-\sqrt{E}), \\ \frac{1}{\sqrt{E'}}\frac{\partial\sqrt{G'}}{\partial u}+\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} &= \frac{\alpha}{\lambda}(\sqrt{G'}+\sqrt{G}). \end{aligned}$$

When these expressions are substituted in the Gauss equation for Σ', analogous to (7), we are led to the condition

$$(23)\qquad 2\sqrt{E'G'}-\sqrt{E'}(l-\sqrt{G})+\sqrt{G'}(k-\sqrt{E})+\sqrt{E}l+\sqrt{G}k=0.$$

If we choose the constant m' of the new transformation R equal to m, it follows from the foregoing results and (6) that

$$(24)\qquad \sigma'=\rho\sigma.$$

In order that μ', α', β', and σ', as given by (17), (19), and (24), shall satisfy equations for Σ' analogous to (I) we must have

$$(25)\qquad \begin{aligned} k'-k+\sqrt{E}-\sqrt{E'} &= 0, \\ l'+l-\sqrt{G}-\sqrt{G'} &= 0. \end{aligned}$$

Moreover, these equations satisfy equations similar to the last two of (I) when (23) is satisfied.

3. Equations in reduced form. The inverse of a transformation R

Equation (23) may be replaced by the two equations

$$(26)\qquad \begin{aligned} \sqrt{E'} &= \frac{1}{2}(\sqrt{E}-k)+\frac{e^{\theta}}{2}(\sqrt{E}+k), \\ -\sqrt{G'} &= \frac{1}{2}(\sqrt{G}-l)+\frac{e^{-\theta}}{2}(\sqrt{G}+l), \end{aligned}$$

where θ is a function to be determined by the equation (22). Reviewing the preceding discussion, we note that when such a function θ is known, the surface Σ' is defined intrinsically by (26) and (21).

When the values (26) of $\sqrt{E'}$ and $\sqrt{G'}$ are substituted in (22), we get for the determination of θ

$$(27)\quad \begin{aligned} (\sqrt{E}+k)e^{\theta}\frac{\partial\theta}{\partial v} &= (e^{\theta}-e^{-\theta})(\sqrt{G}+l)\left(\frac{\beta}{2\lambda}(\sqrt{E}+k)-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right),\\ (\sqrt{G}+l)e^{-\theta}\frac{\partial\theta}{\partial u} &= (e^{\theta}-e^{-\theta})(\sqrt{E}+k)\left(\frac{\alpha}{2\lambda}(\sqrt{G}+l)-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right). \end{aligned}$$

These equations are satisfied by $e^{\theta}=1$. In this case Σ' is congruent to Σ. Another constant solution is $e^{\theta}=-1$. Now equations (26) become

$$(28)\qquad \sqrt{E'}=-k,\qquad \sqrt{G'}=l.$$

Comparing these equations with (11) and (21) with (12), we see that Σ' is Σ_1, the transform of Σ, and consequently its transform Σ_1' is Σ. By means of the foregoing results we shall then be able to find the expressions for the functions λ^{-1}, μ^{-1}, α^{-1}, β^{-1}, σ^{-1} by which Σ is obtained from Σ_1. In fact, from (20) we have, to within a negligible constant factor, $\rho=1/\lambda\sigma$. Consequently from (17), (19), and (24) we have

$$(29)\quad \lambda^{-1}=\frac{1}{\sigma},\qquad \mu^{-1}=\frac{\mu}{\lambda\sigma},\qquad \alpha^{-1}=\frac{\alpha}{\lambda\sigma},\qquad \beta^{-1}=-\frac{\beta}{\lambda\sigma},\qquad \sigma^{-1}=\frac{1}{\lambda}.$$

It is readily shown that these expressions satisfy equations for Σ_1 analogous to (I).

4. Solution of equations (27)

In view of the foregoing results, we are concerned with the solutions of equations (27) other than $e^{2\theta}=1$. By means of (I) these equations are reducible to

$$(30)\qquad \frac{\partial\theta}{\partial u}=(1-e^{2\theta})\frac{\partial B}{\partial u},\qquad \frac{\partial\theta}{\partial v}=-(1-e^{-2\theta})\frac{\partial A}{\partial v},$$

where

$$(31)\qquad A=\log\sqrt{\frac{\sigma}{\lambda}}(\sqrt{E}+k),\qquad B=\log\sqrt{\frac{\sigma}{\lambda}}(\sqrt{G}+l).$$

The condition of integrability of (30) is reducible to

$$(32)\qquad e^{2\theta}L-M=0,$$

where

$$(33)\qquad L=\frac{\partial^2 B}{\partial u\partial v}-2\frac{\partial A}{\partial v}\frac{\partial B}{\partial u},\qquad M=\frac{\partial^2 A}{\partial u\partial v}-2\frac{\partial A}{\partial v}\frac{\partial B}{\partial u}.$$

If $L=M=0$, there are an infinity of deformations of the kind sought.

Bianchi has solved this problem and hence we exclude it from our consideration.

Substituting the expression for $e^{2\theta}$ from (32) in (30), we get

$$\frac{\partial}{\partial u}\left(\frac{L}{M}\right) = 2\left(1 - \frac{L}{M}\right)\frac{\partial B}{\partial u}, \qquad \frac{\partial}{\partial v}\left(\frac{M}{L}\right) = 2\left(1 - \frac{M}{L}\right)\frac{\partial A}{\partial v}.$$

The integrals of these equations are

$$\frac{L}{M} = 1 - Ve^{-2B}, \qquad \frac{M}{L} = 1 - Ue^{-2A},$$

where U and V are functions of u and v respectively.

Since we have excluded the case $e^{2\theta} = 1$, neither U nor V can be zero. The consistency of these equations necessitates

$$UV - Ue^{2B} - Ve^{2A} = 0,$$

which in consequence of (31) is equivalent to

$$\frac{\lambda}{\sigma} UV - (\sqrt{G} + l)^2 U - (\sqrt{E} + k)^2 V = 0.$$

Because of the significance of k and l as shown by (11), we know that the parameters u and v of the lines of curvature on Σ can be chosen so that U and V may be replaced by constants, which it is convenient to take as ± 4. Hence we have for consideration the three cases

$$(34) \qquad (\sqrt{E} + k)^2 + (\sqrt{G} + l)^2 = \pm 4\frac{\lambda}{\sigma}, \qquad U = V = \pm 4,$$

$$(35) \qquad (\sqrt{E} + k)^2 - (\sqrt{G} + l)^2 = 4\frac{\lambda}{\sigma}, \qquad U = -V = 4,$$

$$(36) \qquad (\sqrt{E} + k)^2 - (\sqrt{G} + l)^2 = -4\frac{\lambda}{\sigma}, \qquad U = -V = -4.$$

If we restrict our consideration to the case of real transformations of real surfaces, we may replace (34) by

$$(37) \qquad (\sqrt{E} + k) = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\cos\omega, \qquad (\sqrt{G} + l) = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\sin\omega,$$

where ω is a function to be determined and $\epsilon = \pm 1$ according as λ/σ is positive or negative. Now (31) become

$$A = \log 2\sqrt{\epsilon}\cos\omega, \qquad B = \log 2\sqrt{\epsilon}\sin\omega,$$

and from (32) we get

$$e^{2\theta} = -\tan^2\omega.$$

These values of A and B satisfy (32) when L and M are given the values (33). Since θ is necessarily imaginary, equation (34) does not give rise to a real solution of the problem.

Before considering equation (36), we remark that if we put

$$\bar{\alpha} = \alpha, \qquad \bar{\beta} = \beta, \qquad \bar{\mu} = \mu, \qquad \bar{\lambda} = \lambda, \qquad \bar{\sigma} = -\sigma, \qquad \bar{m} = -m,$$

equations (I) and (6) are satisfied, and from (5) it follows that the transform $\bar{\Sigma}_1$ is the same as Σ_1. Hence equation (36) offers nothing different from (35), which we now proceed to investigate.

If we replace (35) by

$$\text{(38)} \qquad \sqrt{E} + k = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\cosh\omega, \qquad \sqrt{G} + l = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\sinh\omega,$$

where $\epsilon = \pm 1$ according as λ/σ is positive or negative, we have

$$A = \log 2\sqrt{\epsilon}\cosh\omega, \qquad B = \log 2\sqrt{\epsilon}\sinh\omega,$$

and

$$\text{(39)} \qquad e^{2\theta} = \tanh^2\omega.$$

From (I) we obtain

$$\text{(40)} \qquad \frac{\partial \log \lambda\sigma}{\partial u} = (\sqrt{E} + k)\frac{\alpha}{\lambda}, \qquad \frac{\partial \log \lambda\sigma}{\partial v} = (\sqrt{G} + l)\frac{\beta}{\lambda},$$

and

$$\text{(41)} \qquad \begin{aligned} \frac{\partial}{\partial u}\log\sqrt{\frac{\sigma}{\lambda}}(\sqrt{G} + l) &= \frac{\sqrt{E}+k}{\sqrt{G}+l}\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} - \frac{1}{2}(\sqrt{E}+k)\frac{\alpha}{\lambda}, \\ \frac{\partial}{\partial v}\log\sqrt{\frac{\sigma}{\lambda}}(\sqrt{E} + k) &= \frac{\sqrt{G}+l}{\sqrt{E}+k}\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{1}{2}(\sqrt{G}+l)\frac{\beta}{\lambda}. \end{aligned}$$

Substituting the expressions from (38), we get

$$\text{(42)} \qquad \begin{aligned} \frac{\partial\omega}{\partial u} &= \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} - \frac{\alpha}{\sqrt{\lambda\sigma\epsilon}}\sinh\omega, \\ \frac{\partial\omega}{\partial v} &= \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{\beta}{\sqrt{\lambda\sigma\epsilon}}\cosh\omega. \end{aligned}$$

By means of (I) and (40) we find that the condition of integrability of these equations reduces to

$$\text{(43)} \qquad \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right).$$

We shall now determine under what conditions a surface satisfies (43).

5. Surfaces with the same spherical representation of their lines of curvature as isothermic surfaces

If the linear element of the spherical representation of Σ be written

$$d\sigma^2 = \mathcal{E}du^2 + \mathcal{G}dv^2,$$

then

$$\mathcal{E} = \rho_1^2 E, \qquad \mathcal{G} = \rho_2^2 G, \tag{44}$$

as follows from (1). Hence the Codazzi equations (7) may be written

$$\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}, \qquad \frac{1}{\sqrt{\mathcal{E}}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u},$$

and consequently equation (43) may be replaced by

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{\mathcal{E}}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u}\right). \tag{45}$$

It is our purpose to show that an orthogonal system on the unit sphere satisfying this condition represents the lines of curvature of two isothermic surfaces, which are the *Christoffel transforms* of one another by definition.

In the case of an isothermic surface, equation (44) may be replaced by

$$\rho_1 = \frac{a}{\sqrt{\mathcal{E}}}, \qquad \rho_2 = \frac{a}{\sqrt{\mathcal{G}}}, \tag{46}$$

or

$$\rho_1 = \frac{b}{\sqrt{\mathcal{E}}}, \qquad \rho_2 = -\frac{b}{\sqrt{\mathcal{G}}}, \tag{47}$$

where

$$E = G = a^2, \quad \text{or} \quad b^2. \tag{48}$$

The Codazzi equations (7) may be given the form

$$\frac{\partial \log\sqrt{\mathcal{E}}}{\partial v} = \frac{1}{\rho_2 - \rho_1}\frac{\partial\rho_1}{\partial v}, \qquad \frac{\partial \log\sqrt{\mathcal{G}}}{\partial u} = \frac{1}{\rho_1 - \rho_2}\frac{\partial\rho_2}{\partial u}.$$

Substituting in these equations from (46) and (47), we have respectively

$$\begin{aligned} \frac{\partial \log a}{\partial v} &= \frac{1}{\sqrt{\mathcal{G}}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v}, \qquad & \frac{\partial \log a}{\partial u} &= \frac{1}{\sqrt{\mathcal{E}}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u}, \\ \frac{\partial \log b}{\partial v} &= -\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial\sqrt{\mathcal{E}}}{\partial v}, \qquad & \frac{\partial \log b}{\partial u} &= -\frac{1}{\sqrt{\mathcal{E}}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u}. \end{aligned} \tag{49}$$

Equation (45) expresses the necessary and sufficient condition that these equations shall be consistent. Conversely, when the spherical representation of the lines of curvature of a surface satisfies (45), one obtains by quadratures from (48) and (49) the first fundamental coefficients of two isothermic surfaces with the same spherical representation of their lines of curvature as the given surface. Hence we have

THEOREM 2. *The necessary and sufficient condition that a surface be isothermic or have the same spherical representation of its lines of curvature as an isothermic surface is that condition* (43) *be satisfied.*

6. TRANSFORMATIONS R OF THE TYPE SOUGHT

Let Σ be a surface satisfying equation (43). Then equations (I) may be replaced by

$$
\begin{gathered}
\frac{\partial\lambda}{\partial u} = \sqrt{E}\alpha, \qquad \frac{\partial\lambda}{\partial v} = \sqrt{G}\beta, \\
\frac{\partial\mu}{\partial u} = -\frac{\sqrt{E}}{\rho_1}\alpha, \qquad \frac{\partial\mu}{\partial v} = -\frac{\sqrt{G}}{\rho_2}\beta, \\
\frac{\partial\alpha}{\partial u} = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\beta + \mu\frac{\sqrt{E}}{\rho_1} + 2m\sqrt{\lambda\sigma\epsilon}\cosh\omega, \\
\frac{\partial\alpha}{\partial v} = \frac{\beta}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}, \qquad \frac{\partial\beta}{\partial u} = \frac{\alpha}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}, \\
\frac{\partial\beta}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha + \mu\frac{\sqrt{G}}{\rho_2} + 2m\sqrt{\lambda\sigma\epsilon}\sinh\omega, \\
\frac{\partial\log\sigma}{\partial u} = \left(2\sqrt{\frac{\lambda\epsilon}{\sigma}}\cosh\omega - \sqrt{E}\right)\frac{\alpha}{\lambda}, \\
\frac{\partial\log\sigma}{\partial v} = \left(2\sqrt{\frac{\lambda\epsilon}{\sigma}}\sinh\omega - \sqrt{G}\right)\frac{\beta}{\lambda}, \\
\frac{\partial\omega}{\partial u} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} - \frac{\alpha}{\sqrt{\lambda\sigma\epsilon}}\sinh\omega, \\
\frac{\partial\omega}{\partial v} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{\beta}{\sqrt{\lambda\sigma\epsilon}}\cosh\omega.
\end{gathered} \tag{50}
$$

In consequence of these equations, the expressions for k and l of the surface Σ_1, namely

$$
k = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\cosh\omega - \sqrt{E}, \qquad l = 2\sqrt{\frac{\lambda\epsilon}{\sigma}}\sinh\omega - \sqrt{G}, \tag{51}
$$

satisfy the last two of equations (I). Moreover, it is readily seen that

$$
\frac{\partial}{\partial u}\left(\frac{1}{l}\frac{\partial k}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{1}{k}\frac{\partial l}{\partial u}\right),
$$

so that Σ_1 is a surface satisfying (43), as was evident from general considerations.

From (26) and (39) we have

$$(52) \qquad \sqrt{E'} = \sqrt{E} - \sqrt{\frac{\lambda\epsilon}{\sigma}}\, e^{-\omega}, \qquad -\sqrt{G'} = \sqrt{G} + \sqrt{\frac{\lambda\epsilon}{\sigma}}\, e^{-\omega},$$

and from (25)

$$(53) \qquad k' = \sqrt{\frac{\lambda\epsilon}{\sigma}}\, e^{\omega} - \sqrt{E}, \qquad -l' = \sqrt{\frac{\lambda\epsilon}{\sigma}}\, e^{\omega} - \sqrt{G}.$$

Equations (22) reduce to

$$\frac{1}{\sqrt{G'}}\frac{\partial\sqrt{E'}}{\partial v} + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{\beta}{\sqrt{\lambda\sigma\epsilon}}\, e^{-\omega} = 0,$$

$$\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{G'}}{\partial u} + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} + \frac{\alpha}{\sqrt{\lambda\sigma\epsilon}}\, e^{-\omega} = 0,$$

from which it readily follows that E' and G' satisfy (43),

Equations (21) become

$$(54) \qquad \frac{\sqrt{E'}}{\rho_1'} = \frac{\sqrt{E}}{\rho_1} + \frac{\mu}{\sqrt{\lambda\sigma\epsilon}}\, e^{-\omega}, \qquad -\frac{\sqrt{G'}}{\rho_2'} = \frac{\sqrt{G}}{\rho_2} - \frac{\mu}{\sqrt{\lambda\sigma\epsilon}}\, e^{-\omega}.$$

From (2) it follows that if a transformation R is determined by functions λ and μ, another $\bar{R}$, is given by $\lambda + c$ and μ, where c is any constant. Denoting the functions for $\bar{R}$ by $\bar{\lambda}, \bar{\mu}, \cdots, \bar{\sigma}$, we have from (I)

$$(55) \qquad \bar{\lambda} = \lambda + c, \qquad \bar{\mu} = \mu, \qquad \bar{\alpha} = \alpha, \qquad \bar{\beta} = \beta, \qquad \bar{m} = m,$$

and from (6)

$$\bar{\lambda}\bar{\sigma} = \lambda\sigma, \qquad \bar{\sigma} = \frac{\lambda\sigma}{\lambda + c}.$$

Substituting these values in (I), we find that

$$\bar{k} = \frac{1}{\lambda}\left(k(\lambda + c) + c\sqrt{E}\right), \qquad \bar{l} = \frac{1}{\lambda}\left(l(\lambda + c) + c\sqrt{G}\right).$$

If $\bar{\Sigma}_1$ denotes the corresponding transform of Σ, it follows from (10) that Σ_1 and $\bar{\Sigma}_1$ correspond with parallelism of tangent planes.

For any transformation R the tangent planes to S are normal to the lines joining corresponding points on Σ and Σ_1. From (5) and (55) it follows that S and $\bar{S}$, the surface of centers of the transformation $\bar{R}$, correspond with parallelism of tangent planes.

When these results are applied to the type of transformations R given by (50) we have

$$\bar{\omega} = \omega.$$

7. When Σ is an isothermic surface

From (2) it follows that μ satisfies the tangential equation of Σ, that is the equation satisfied by the direction-cosines of the normal to Σ. Hence if $\bar{\Sigma}$ is any surface having the same spherical representation of its lines of curvature as Σ, the function μ and the function $\bar{\lambda}$, given by equations for $\bar{\Sigma}$ analogous to (2), determine a transformation R for $\bar{\Sigma}$. We say that this transformation of $\bar{\Sigma}$ is obtained from the given transformation R of Σ by a *transformation of Combescure.* From this result it is clear that if we find all the transformations R of the type (50) of isothermic surfaces, all the other solutions of our problem are obtainable from these by transformations of Combescure. Accordingly we confine ourselves hereafter to the case where Σ is an isothermic surface.

When Σ is isothermic, the parameters of the lines of curvature may be chosen so that

$$\sqrt{E} = \sqrt{G} = e^{\phi}. \tag{56}$$

Now

$$k + l = 2\left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - e^{\phi}\right). \tag{57}$$

From equations (50) we derive the equations

$$\frac{\partial}{\partial u}\left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - e^{\phi}\right) = \left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - e^{\phi}\right)\left(\frac{\partial\phi}{\partial u} - \frac{\alpha e^{\omega}}{\sqrt{\lambda\sigma\epsilon}}\right),$$

$$\frac{\partial}{\partial v}\left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - e^{\phi}\right) = \left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - e^{\phi}\right)\left(\frac{\partial\phi}{\partial v} - \frac{\beta e^{\omega}}{\sqrt{\lambda\sigma\epsilon}}\right),$$

and the last two of equations (50) reduce to

$$\frac{\partial\phi}{\partial u} - \frac{\alpha e^{\omega}}{\sqrt{\lambda\sigma\epsilon}} = \frac{\partial\omega}{\partial u} - \frac{\partial \log\sqrt{\lambda\sigma}}{\partial u},$$

$$\frac{\partial\phi}{\partial v} - \frac{\beta e^{\omega}}{\sqrt{\lambda\sigma\epsilon}} = \frac{\partial\omega}{\partial v} - \frac{\partial \log\sqrt{\lambda\sigma}}{\partial v}.$$

The integral of these equations is

$$e^{\omega} = \frac{\sqrt{\lambda\sigma\epsilon}\,e^{\phi}}{\lambda + c}, \tag{58}$$

where c denotes a constant of integration.

8. TRANSFORMATIONS D_m OF Σ. ISOTHERMIC SURFACES PARALLEL TO Σ'

When c in (58) is zero, equations (50) become

$$(59)\quad \begin{aligned} &\frac{\partial\lambda}{\partial u} = e^{\phi}\,\alpha, \qquad \frac{\partial\lambda}{\partial v} = e^{\phi}\,\beta, \\ &\frac{\partial\mu}{\partial u} = -\frac{e^{\phi}}{\rho_1}\alpha, \qquad \frac{\partial\mu}{\partial v} = -\frac{e^{\phi}}{\rho_2}\beta, \\ &\frac{\partial\sigma}{\partial u} = e^{-\phi}\,\alpha, \qquad \frac{\partial\sigma}{\partial v} = -\,e^{-\phi}\,\beta, \\ &\frac{\partial\alpha}{\partial u} = -\frac{\partial\phi}{\partial v}\beta + \mu\frac{e^{\phi}}{\rho_1} + m\,(\sigma e^{\phi} + \lambda e^{-\phi}), \\ &\frac{\partial\alpha}{\partial v} = \beta\frac{\partial\phi}{\partial u}, \qquad \frac{\partial\beta}{\partial u} = \alpha\frac{\partial\phi}{\partial v}, \\ &\frac{\partial\beta}{\partial v} = -\frac{\partial\phi}{\partial u}\alpha + \mu\frac{e^{\phi}}{\rho_2} + m\,(\sigma e^{\phi} - \lambda e^{-\phi}), \end{aligned}$$

and from (51) we have

$$(60)\qquad k = -\,l = \frac{\lambda}{\sigma}e^{-\phi}.$$

These are the equations of transformations D_m of isothermic surfaces into isothermic surfaces discovered by Darboux.*

From (52) and (53) we obtain

$$(61)\qquad \sqrt{E'} = e^{\phi} - \frac{\lambda}{\sigma}e^{-\phi}, \qquad -\sqrt{G'} = e^{\phi} + \frac{\lambda}{\sigma}e^{-\phi},$$

and

$$(62)\qquad k' = l' = 0.$$

Hence Σ_1' is a point and not a surface.

Conversely, if a transformation R can be deformed in such a way that one of the sheets of the new envelope is a point, the requirement that the lines of curvature on the two sheets of the new envelope shall correspond is satisfied identically and consequently the preceding formulas apply. For this case it follows from (53) that

$$\sqrt{E} = \sqrt{G} = \sqrt{\frac{\lambda\epsilon}{\sigma}}\,e^{\omega},$$

and consequently the transformation R is a D_m.†

*Annales de l'École Normale Supérieure, ser. 3, vol. 16 (1899), pp. 491–508.

† Cf. Eisenhart, Rendiconti dei Lincei, l. c.

For the present case equations (20) become

$$\frac{\partial \log \rho}{\partial u}+\frac{\alpha}{\sigma}e^{-\phi}=0,\qquad \frac{\partial \log \rho}{\partial v}-\frac{\beta}{\sigma}e^{-\phi}=0,$$

from which it follows that $\rho\sigma$ is constant. Since the functions of a transformation R are determined only to within a constant factor, it follows from § 2 that the functions by means of which Σ' is transformed into the point Σ_1' are given by

$$\alpha'=\frac{\alpha}{\sigma},\qquad \beta'=-\frac{\beta}{\sigma},\qquad \mu'=\frac{\mu}{\sigma},\qquad \lambda'=\frac{\lambda}{\sigma},\qquad \sigma'=1. \tag{63}$$

From the general results of this paper it follows that Σ' has the same spherical representation of its lines of curvature as two isothermic surfaces, say $\overline{\Sigma}$ and $\overline{\Sigma}'$. From (54) it follows that this spherical representation is determined by

$$\sqrt{\mathcal{E}'}=\frac{\sqrt{E'}}{\rho_1'}=\frac{e^{\phi}}{\rho_1}+\frac{\mu e^{-\phi}}{\sigma},\qquad \sqrt{\mathcal{G}'}=\frac{\sqrt{G'}}{\rho_2'}=-\frac{e^{\phi}}{\rho_2}+\frac{\mu}{\sigma}e^{-\phi}. \tag{64}$$

Applying the results of § 5, we find that the first fundamental coefficients of $\overline{\Sigma}$ and $\overline{\Sigma}'$ are given by

$$\sqrt{\overline{E}}=\sqrt{\overline{G}}=\sigma e^{-\phi},\qquad \sqrt{\overline{E}'}=\sqrt{\overline{G}'}=e^{-\phi}/\sigma. \tag{65}$$

The principal radii of normal curvature of these surfaces are given by

$$\begin{aligned}&\frac{1}{\overline{\rho}_1}=\frac{e^{-\phi}}{\sigma}\left(\frac{e^{\phi}}{\rho_1}+\frac{\mu}{\sigma}e^{-\phi}\right),\qquad \frac{1}{\overline{\rho}_2}=\frac{e^{-\phi}}{\sigma}\left(\frac{e^{\phi}}{\rho_2}-\frac{\mu}{\sigma}e^{-\phi}\right),\\ &\frac{1}{\overline{\rho}_1'}=e^{\phi}\sigma\left(\frac{e^{\phi}}{\rho_1}+\frac{\mu}{\sigma}e^{-\phi}\right),\qquad \frac{1}{\overline{\rho}_2'}=-e^{\phi}\sigma\left(\frac{e^{\phi}}{\rho_2}-\frac{\mu}{\sigma}e^{-\phi}\right).*\end{aligned} \tag{66}$$

9. Transformations D_m of $\overline{\Sigma}$ and $\overline{\Sigma}'$, and their deformations

Since $\overline{\Sigma}$ and $\overline{\Sigma}'$ have the same spherical representation of their lines of curvature as Σ', a transformation R of each of these surfaces is given by taking for μ the value of μ' in (63). From the third and fourth of equations (I) we see that the corresponding functions α and β differ at most in signs from α' and β' as given by (63). We consider the two surfaces separately.

For $\overline{\Sigma}$ we find that the functions are given by

$$\overline{\alpha}=\frac{\alpha}{\sigma},\qquad \overline{\beta}=\frac{\beta}{\sigma},\qquad \mu=\frac{\mu}{\sigma},\qquad \overline{\lambda}=\lambda+c,$$

$$\overline{\sigma}=\frac{-\lambda}{\sigma(\lambda+c)},\qquad \overline{m}=-m,$$

* Evidently for the surfaces symmetric to $\overline{\Sigma}$ and $\overline{\Sigma}'$ with respect to a point, the expressions analogous to (65) are the same and to (66) differ in sign.

where c is any constant. Furthermore,

$$\bar{k} = \left(ce^{\phi} \frac{\sigma}{\lambda} - (\lambda + c) e^{-\phi} \right), \qquad l = \left(ce^{\phi} \frac{\sigma}{\lambda} + (\lambda + c) e^{-\phi} \right).$$

When $c = 0$, the surface $\bar{\Sigma}_1$ is isothermic and the transformation is a D_m. The values of the functions for this case are

$$\alpha = \frac{\alpha}{\sigma}, \quad \bar{\beta} = \frac{\beta}{\sigma}, \quad \bar{\mu} = \frac{\mu}{\sigma}, \quad \bar{\lambda} = \lambda, \quad \bar{\sigma} = -\frac{1}{\sigma}. \tag{67}$$

From equations analogous to (12) we find for the principal radii of $\bar{\Sigma}_1$ the expressions

$$\frac{1}{\rho_{11}} = \frac{e^{2\phi}}{\lambda} \left(\frac{1}{\rho_1} + \frac{\mu}{\lambda} \right), \qquad \frac{1}{\bar{\rho}_{12}} = \frac{e^{2\phi}}{\lambda} \left(\frac{1}{\rho_2} + \frac{\mu}{\lambda} \right). \tag{68}$$

In a similar manner, transformations of $\bar{\Sigma}'$ are given by

$$\alpha' = \frac{\alpha}{\sigma}, \quad \bar{\beta}' = -\frac{\beta}{\sigma}, \quad \bar{\mu}' = \frac{\mu}{\sigma}, \quad \bar{\lambda}' = -\frac{1}{\sigma},$$

$$\bar{\sigma}' = \frac{\lambda}{1 - c\sigma}, \qquad \bar{m}' = -m,$$

$$\bar{k}' = \left(e^{\phi} \frac{c\sigma - 1}{\lambda} - ce^{-\phi} \right), \qquad \bar{l}' = -\left(e^{\phi} \frac{c\sigma - 1}{\lambda} + ce^{-\phi} \right).$$

In particular a transformation D_{-m} of $\bar{\Sigma}'$ is given by

$$\alpha' = \frac{\alpha}{\sigma}, \quad \bar{\beta}' = -\frac{\beta}{\sigma}, \quad \bar{\mu}' = \frac{\mu}{\sigma}, \quad \bar{\lambda}' = -\frac{1}{\sigma}, \quad \bar{\sigma}' = \lambda, \tag{69}$$

and the fundamental functions for the transform $\bar{\Sigma}_1'$ have the expressions

$$\begin{gathered} \bar{k}' = -\bar{l}' = -e^{\phi}/\lambda, \\ \frac{1}{\bar{\rho}_{11}'} = \mu + \frac{\lambda}{\rho_1}, \qquad \frac{1}{\bar{\rho}_{12}'} = \mu + \frac{\lambda}{\rho_2}. \end{gathered} \tag{70}$$

The isothermic surface defined by (70) has been considered by Bianchi;* he calls the process by which it is obtained from Σ and the functions of a transformation D_m of Σ the transformation T_m determined by the given D_m. This transformation was defined intrinsically by Bianchi without any indication of its geometrical relation to Σ.

We shall now apply the results of § 8 to these surfaces $\bar{\Sigma}$ and $\bar{\Sigma}'$, by deforming the surfaces of center $\bar{S}$ and $\bar{S}'$ of the transformations D_{-m} of these surfaces defined by (67) and (69).

* Annali di matematica, ser. 3, vol. 12 (1906), pp. 19–54.

We call (Σ') the surface analogous to Σ'. From (61) and (64) we find that its fundamental functions are given by

$$(\sqrt{E'}) = (\sigma e^{\phi} + \lambda e^{-\phi}), \qquad (\sqrt{G'}) = -(\sigma e^{\phi} - \lambda e^{-\phi}),$$

$$\left(\frac{\sqrt{E'}}{\rho_1'}\right) = \frac{e^{\phi}}{\rho_1}, \qquad \left(\frac{\sqrt{G'}}{\rho_2'}\right) = -\frac{e^{\phi}}{\rho_2}.$$

Hence (Σ') has the same spherical representation of its lines of curvature as Σ_1. Furthermore, from equations analogous to (65) and (66), we find that the surfaces $(\overline{\Sigma})$, $(\overline{\Sigma}')$ analogous to $\overline{\Sigma}$, $\overline{\Sigma}'$ are defined by

$$(\sqrt{\overline{E}}) = (\sqrt{\overline{G}}) = e^{\phi}, \qquad \left(\frac{1}{\overline{\rho}_1}\right) = -\frac{1}{\rho_1}, \qquad \left(\frac{1}{\overline{\rho}_2}\right) = -\frac{1}{\rho_2},$$

$$(\sqrt{\overline{E}'}) = (\sqrt{\overline{G}'}) = e^{-\phi}, \qquad \left(\frac{1}{\overline{\rho}_1'}\right) = -\frac{e^{2\phi}}{\rho_1}, \qquad \left(\frac{1}{\overline{\rho}_2'}\right) = \frac{e^{2\phi}}{\rho_2}.$$

Hence $(\overline{\Sigma})$ and (Σ') are respectively the symmetric of Σ with respect to a point and its Christoffel transform.

Proceeding in like manner with $\overline{\Sigma}'$, we get for $(\Sigma')'$, the surface analogous to Σ', the functions

$$(\sqrt{E'})' = \frac{1}{\lambda\sigma}(\sigma e^{\phi} + \lambda e^{-\phi}), \qquad (\sqrt{G'})' = \frac{1}{\lambda\sigma}(\sigma e^{\phi} - \lambda e^{-\phi}),$$

$$\left(\frac{\sqrt{E'}}{\rho_1'}\right)' = \left(\frac{e^{\phi}}{\rho_1} + \frac{\mu}{\sigma}e^{-\phi} + \frac{\mu}{\lambda}e^{\phi}\right), \qquad \left(\frac{\sqrt{G'}}{\rho_2'}\right)' = -\left(\frac{e^{\phi}}{\rho_2} + \frac{\mu}{\sigma}e^{-\phi} - \frac{\mu}{\lambda}e^{\phi}\right).$$

This surface has the same spherical representation of its lines of curvature as Σ_1. The surfaces $(\overline{\Sigma})'$ and $(\overline{\Sigma}')'$ are respectively the symmetric of Σ_1 and the Christoffel transform.

Let S_0 be the surface of centers of the spheres enveloped by Σ' and Σ_1', the latter being a point, say O. Corresponding normals to Σ', $\overline{\Sigma}$, and $\overline{\Sigma}'$ are parallel. In like manner, the normals to $\overline{\Sigma}_1$ and $\overline{\Sigma}_1'$ are parallel to the lines joining O to corresponding points of S_0. Hence the permanent conjugate system on S_0, that is the system corresponding to the lines of curvature on Σ, project upon the unit sphere with center O into the orthogonal system representing the lines of curvature on $\overline{\Sigma}_1$ and $\overline{\Sigma}_1'$. Applying the same reasoning to the transformation resulting from the deformation of $\overline{S}$, the surface of centers of the spheres enveloped by $\overline{\Sigma}'$ and $\overline{\Sigma}_1'$, we see that the lines joining O to points of $\overline{S}_0$, the deform of $\overline{S}$, are parallel to the normals to Σ. Furthermore, it is readily seen that $\overline{S}$ and S_0 correspond with parallelism of tangent planes; and likewise S and $\overline{S}_0$.*

* The results of this paragraph were obtained by Bianchi by purely analytical processes as a result of his intrinsic definition of the transformation T_m. Cf. Rendiconti della R. Accademia dei Lincei, ser. 3, vol. 24 (1915), p. 386.

10. Case $c \neq 0$ in equation (58)

When we substitute the value (58) of ω with $c \neq 0$ in equations (50), they become

$$
(71)\quad
\begin{aligned}
&\frac{\partial\lambda}{\partial u} = e^{\phi}\alpha, \qquad \frac{\partial\lambda}{\partial v} = e^{\phi}\beta,\\
&\frac{\partial\mu}{\partial u} = -\frac{e^{\phi}}{\rho_1}\alpha, \qquad \frac{\partial\mu}{\partial v} = -\frac{e^{\phi}}{\rho_2}\beta,\\
&\frac{\partial\sigma}{\partial u} = \frac{\alpha}{\lambda}\left(\frac{-ce^{\phi}\sigma}{\lambda+c} + (\lambda+c)e^{-\phi}\right),\\
&\frac{\partial\sigma}{\partial v} = -\frac{\beta}{\lambda}\left(\frac{ce^{\phi}\sigma}{\lambda+c} + (\lambda+c)e^{-\phi}\right),\\
&\frac{\partial\alpha}{\partial u} = -\frac{\partial\phi}{\partial v}\beta + \mu\frac{e^{\phi}}{\rho_1} + m\left(\frac{\lambda\sigma}{\lambda+c} + (\lambda+c)e^{-\phi}\right),\\
&\frac{\partial\alpha}{\partial v} = \beta\frac{\partial\phi}{\partial u}, \qquad \frac{\partial\beta}{\partial u} = \alpha\frac{\partial\phi}{\partial v},\\
&\frac{\partial\beta}{\partial v} = -\frac{\partial\phi}{\partial u}\alpha + \mu\frac{e^{\phi}}{\rho_2} + m\left(\frac{\lambda\sigma}{\lambda+c} - (\lambda+c)e^{-\phi}\right),
\end{aligned}
$$

and

$$
(72)\quad k = -\frac{ce^{\phi}}{\lambda+c} + \frac{\lambda+c}{\sigma}e^{-\phi}, \qquad l = -\frac{ce^{\phi}}{\lambda+c} - \frac{\lambda+c}{\sigma}e^{-\phi}.
$$

For the surface Σ' we have the functions

$$
(73)\quad
\begin{aligned}
&\sqrt{E'} = e^{\phi} - \frac{\lambda+c}{\sigma}e^{-\phi}, \qquad -\sqrt{G'} = e^{\phi} + \frac{\lambda+c}{\sigma}e^{-\phi},\\
&\frac{\sqrt{E'}}{\rho_1'} = \frac{e^{\phi}}{\rho_1} + \frac{\mu}{\lambda}\frac{\lambda+c}{\sigma}e^{-\phi}, \qquad \frac{\sqrt{G'}}{\rho_2'} = -\frac{e^{\phi}}{\rho_2} + \frac{\mu}{\lambda}\frac{\lambda+c}{\sigma}e^{-\phi},
\end{aligned}
$$

and for Σ_1'

$$
(74)\quad
\begin{aligned}
&k' = -l' = -\frac{ce^{\phi}}{\lambda+c},\\
&\frac{1}{\rho_{11}'} = \frac{1}{c}\left(\frac{\lambda+c}{\rho_1} + \mu\right), \qquad \frac{1}{\rho_{12}'} = \frac{1}{c}\left(\frac{\lambda+c}{\rho_2} + \mu\right).
\end{aligned}
$$

The last two are a consequence of (73) and of equations analogous to (12). From these expressions it is seen that Σ_1' is an isothermic surface. We shall now determine its relation to Σ.

It is readily seen that, if λ, μ, α, β, σ are solutions of equations (71), a set of solutions of (59) is given by

$$(\lambda) = \lambda + c, \qquad (\mu) = \mu, \qquad (\alpha) = \alpha, \qquad (\beta) = \beta,$$
$$(\sigma) = \frac{\lambda\sigma}{\lambda + c}, \qquad (m) = m.$$

Hence these functions define a transformation (D_m) of Σ, and a comparison of (70) and (74) reveals the fact that Σ_1', as given by (74) is homothetic to the transform of Σ by the T_m determined by (D_m).

11. When Σ_1' is a plane

From equations (21), (53), and equations for Σ' analogous to (12), we find the following expressions for the principal radii of normal curvature of Σ_1':

$$(75)\qquad \begin{aligned} &\frac{1}{\rho_{11}'}\left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - \sqrt{E}\right) + \frac{\sqrt{E}}{\rho_1} + \frac{\mu}{\sqrt{\lambda\sigma\epsilon}}e^{\omega} = 0, \\ &\frac{1}{\rho_{12}'}\left(\sqrt{\frac{\lambda\epsilon}{\sigma}}e^{\omega} - \sqrt{G}\right) + \frac{\sqrt{G}}{\rho_2} + \frac{\mu}{\sqrt{\lambda\sigma\epsilon}}e^{\omega} = 0. \end{aligned}$$

Hence the necessary condition that Σ_1' be a plane is

$$(76)\qquad \frac{\sqrt{E}}{\rho_1} = \frac{\sqrt{G}}{\rho_2} = -\frac{\mu}{\sqrt{\lambda\sigma\epsilon}}e^{\omega} = e^{-\theta},$$

where θ is a function thus defined. This condition is also sufficient; otherwise k', l', or both, is zero and the lines of curvature on Σ_1' are minimal.

In consequence of (76) equations (50) become

$$(77)\qquad \begin{aligned} &\frac{\partial\lambda}{\partial u} = \sqrt{E}\alpha, \qquad \frac{\partial\lambda}{\partial v} = \sqrt{G}\beta, \\ &\frac{\partial\mu}{\partial u} = -e^{-\theta}\alpha, \qquad \frac{\partial\mu}{\partial v} = -e^{-\theta}\beta, \\ &\frac{\partial\alpha}{\partial u} = \frac{\partial\theta}{\partial v}\beta + \mu e^{-\theta} - m\left(\frac{\lambda\sigma}{\mu}e^{-\theta} + \mu e^{\theta}\right), \\ &\frac{\partial\alpha}{\partial v} = -\beta\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\beta}{\partial u} = -\alpha\frac{\partial\theta}{\partial v}, \\ &\frac{\partial\beta}{\partial v} = \frac{\partial\theta}{\partial u}\alpha + \mu e^{-\theta} - m\left(\frac{\lambda\sigma}{\mu}e^{-\theta} - \mu e^{\theta}\right), \\ &\frac{\partial}{\partial u}\frac{\sigma\lambda}{\mu} = -\alpha e^{\theta}, \qquad \frac{\partial}{\partial v}\frac{\sigma\lambda}{\mu} = \beta e^{\theta}, \end{aligned}$$

and the last two of equations (50) are satisfied identically. Now

$$(78)\quad k = -\left(e^{-\theta}\frac{\lambda}{\mu} + e^{\theta}\frac{\mu}{\sigma} + \sqrt{E}\right), \qquad l = -\left(e^{-\theta}\frac{\lambda}{\mu} - e^{\theta}\frac{\mu}{\sigma} + \sqrt{G}\right).$$

Making use of (11) and (12), we get

$$(79)\quad \frac{\sqrt{E_1}}{\rho_{11}} = \frac{\sqrt{G_1}}{\rho_{12}} = -e^{\theta}\frac{\mu^2}{\lambda\sigma}.$$

Hence Σ and Σ_1 are surfaces with isothermal spherical representation of their lines of curvature. Equations (77) define the transformations E_m of surfaces with isothermal representation of their lines of curvature into surfaces of the same kind, which we have discussed previously from several points of view.*

From (53) and (54) we get

$$(80)\quad \begin{aligned} \sqrt{E'} &= \frac{\mu e^{\theta}}{\sigma} + \sqrt{E}, \qquad \sqrt{G'} = \frac{\mu e^{\theta}}{\sigma} - \sqrt{G}, \\ \sqrt{\mathcal{E}'} &= \frac{\sqrt{E'}}{\rho_1'} = e^{-\theta} - \frac{\mu^2}{\lambda\sigma}e^{\theta}, \\ \sqrt{\mathcal{G}'} &= \frac{\sqrt{G'}}{\rho_2'} = -e^{-\theta} - \frac{\mu^2}{\lambda\sigma}e^{\theta}, \end{aligned}$$

$\mathcal{E}'$ and $\mathcal{G}'$ being the coefficients of the spherical representation of Σ'. Also

$$(81)\quad k' = -\left(\frac{\lambda}{\mu}e^{-\theta} + \sqrt{E}\right), \qquad l' = \frac{\lambda}{\mu}e^{-\theta} - \sqrt{G},$$

so that the parametric curves on the plane Σ_1' form an orthogonal system. Hence *the permanent conjugate system on* S_0, *the locus of centers of the spheres tangent to* Σ' *and* Σ_1', *projects into an orthogonal system on the plane* Σ_1'.†

12. Isothermic surfaces determined by a transformation E_m

From the general theory of § 5 we know that there are two isothermic surfaces $\bar{\Sigma}$ and $\bar{\Sigma}'$ with the same spherical representation of their lines of curvature as Σ'. We proceed to their determination.

From (80) and (49) we find that the first fundamental coefficients of these surfaces are

$$(82)\quad \sqrt{\bar{E}} = \sqrt{\bar{G}} = \frac{\mu}{\lambda\sigma}e^{\theta}, \qquad \sqrt{\bar{E}'} = \sqrt{\bar{G}'} = \frac{\lambda\sigma}{\mu}e^{-\theta},$$

* Annals of Mathematics, ser. 2, vol. 17 (1915), p. 64.

† Cf. Bianchi, Rendiconti dei Lincei, ser. 5, vol. 24 (1915), p. 378.

and their principal radii have the values

$$
(83)\qquad \begin{aligned}
&\frac{1}{\bar{\rho}_1} = \frac{\lambda\sigma}{\mu} e^{-2\theta} - \mu, &\qquad &\frac{1}{\bar{\rho}_2} = -\frac{\lambda\sigma}{\mu} e^{-2\theta} - \mu,\\
&\frac{1}{\bar{\rho}_1'} = \frac{\mu}{\lambda\sigma} - \frac{\mu^3}{\lambda^2\sigma^2} e^{2\theta}, &\qquad &\frac{1}{\bar{\rho}_2'} = \frac{\mu}{\lambda\sigma} + \frac{\mu^3}{\lambda^2\sigma^2} e^{2\theta}.
\end{aligned}
$$

From (20), (80), and (78) it follows that we may take $\rho = \mu/\lambda\sigma$, so that the transformation R of Σ' into Σ_1' is given by

$$
\alpha' = \frac{\alpha\mu}{\lambda\sigma}, \qquad \beta' = -\frac{\beta\mu}{\lambda\sigma}, \qquad \mu' = \frac{\mu^2}{\lambda\sigma}, \qquad \lambda' = \frac{\mu}{\sigma}, \qquad \sigma' = \frac{\mu}{\lambda}.
$$

The transformations R of $\bar{\Sigma}$ which are the Combescure transforms of the former are defined by

$$
(84)\qquad \begin{aligned}
&\bar{\alpha} = \frac{\alpha\mu}{\lambda\sigma}, \qquad \bar{\beta} = -\frac{\beta\mu}{\lambda\sigma}, \qquad \bar{\mu} = \frac{\mu^2}{\lambda\sigma},\\
&\bar{\lambda} = \frac{\mu}{\lambda\sigma} + c, \qquad \bar{\sigma} = \frac{-\mu}{1 + c\dfrac{\lambda\sigma}{\mu}}, \qquad \bar{m} = -m,
\end{aligned}
$$

where c denotes a constant. From geometrical considerations we know that $\bar{\Sigma}_1$ is a plane; it is referred to an orthogonal system whose coefficients are

$$
(85)\qquad \bar{k} = ce^{\theta} - \frac{e^{-\theta}}{\mu}\left(1 + \frac{c\lambda\sigma}{\mu}\right), \qquad \bar{l} = ce^{\theta} + \frac{e^{-\theta}}{\mu}\left(1 + \frac{c\lambda\sigma}{\mu}\right).
$$

When $c = 0$, the coördinate system on the plane $\bar{\Sigma}_1$ is isothermal. Moreover, the functions (84) with $c = 0$ satisfy equations (59) and determine a transformation D_{-m} of $\bar{\Sigma}$.

In like manner we have transformations R of $\bar{\Sigma}'$ given by

$$
\begin{aligned}
&\bar{\alpha}' = \frac{\alpha\mu}{\lambda\sigma}, \qquad \bar{\beta}' = \frac{\beta\mu}{\lambda\sigma}, \qquad \bar{\mu}' = \frac{\mu^2}{\lambda\sigma},\\
&\bar{\lambda}' = c - \mu, \qquad \bar{\sigma}' = \frac{-\mu^2}{\lambda\sigma(c - \mu)}, \qquad \bar{m}' = -m,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{k}' &= -ce^{-\theta}\frac{\lambda\sigma}{\mu^2} + e^{\theta}(c - \mu),\\
\bar{l}' &= -ce^{-\theta}\frac{\lambda\sigma}{\mu^2} - e^{\theta}(c - \mu).
\end{aligned}
$$

The surface $\bar{\Sigma}_1'$ is a plane, and for $c = 0$ we have a transformation D_{-m} of $\bar{\Sigma}'$.

The isothermic surfaces $\bar{\Sigma}$ and $\bar{\Sigma}'$ belong to the group discussed in § 8. For every transformation E_m has for one of its Combescure transforms a D_m, namely the transformation of the minimal surface having the given isothermal spherical representation of its lines of curvature.

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